

DEFORMATIONS OF CANONICAL TRIPLE COVERS

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ABSTRACT. In this paper, we show that if X is a smooth variety of general type of dimension $m \geq 3$ for which the canonical map induces a triple cover onto Y , where Y is a projective bundle over \mathbf{P}^1 or onto a projective space or onto a quadric hypersurface, embedded by a complete linear series, then the general deformation of the canonical morphism of X is again canonical and induces a triple cover. The extremal case when Y is embedded as a variety of minimal degree is of interest, due to its appearance in numerous situations. This is especially interesting as well, since it has no lower dimensional analogues.

INTRODUCTION

In this article we prove new results on deformations of canonical morphisms of degree 3 of a variety of general type of arbitrary dimension when the image Y of the morphism is a rational variety such as the projective space, a quadric hypersurface or a \mathbf{P}^n bundle over a projective line. We give special emphasis on those Y when it is a variety of minimal degree. This is a new case that does not occur in either curves or surfaces. Note that the degree of the canonical morphism for curves is bounded by 2. It was shown in [GP03] that there are no odd degree canonical covers of smooth surfaces of minimal degree. The geometric genus of a triple canonical cover of a singular surface of minimal degree is bounded by 5. So the geometry of the higher dimensional covers has no analogy in lower dimensions. In this article we handle more general triple covers even when Y is not of minimal degree. It is a well known fact that low degree covers of varieties of minimal degree have a ubiquitous presence in the geometry of varieties of general type, especially in the case of algebraic surfaces and threefolds such as Calabi-Yau. They appear in the geometry of higher dimensional varieties of general type as well, as [Fuj83] and [Kob92] and the very recent work of [CPG15] show. These deal mostly with the case of double covers. The work [Fuj88] deals with structure theorem for triple covers.

The result of Castelnuovo for an algebraic surface of general type says that if the linear system $|K_X|$ is birational, then $K_X^2 \geq 3p_g - 7$. There is no known general result for a threefold of general type in these directions. The triple canonical covers of rational varieties satisfy the inequality $K_X^3 \geq 3p_g - 9$. There is a strong correlation in the numerology between surfaces and threefolds. It was shown in a very recent work in [CPG15] that the geometry of Horikawa threefolds, which satisfy $K_X^3 = 2p_g - 6$ has striking similarities to Horikawa surfaces, that is those surfaces which are on the Noether line $K_X^2 = 2p_g - 4$. This analogy is important for understanding the geometry of the moduli spaces of higher dimensional varieties. The numerology suggests that for a threefold, the analogy of Castelnuovo's bound could be $K_X^3 \geq 3p_g - 9$. That is for a threefold X , if the linear system $|K_X|$ induces a birational map, then $K_X^3 \geq 3p_g - 9$. In any case, the equality $K_X^3 = 3p_g - 9$, is sufficiently high for a threefold, and $|K_X|$ can be potentially (or conjecturally) birational. This is just for the case when the image of the canonical map Y is of minimal degree. If the image is not of minimal degree, then K_X^3 can be much higher. But the result in this article shows that even when K_X^3 is sufficiently high, as in the case of triple covers, the deformation of the the degree 3 morphisms are again degree 3. The impact of these results on the moduli components is clear.

For the case of algebraic surfaces, the analogues of the results proved here for higher dimensions, showed such an impact on the moduli components, as demonstrated in [GGP10].

It is known from results in [GP03], that there are no triple canonical covers of a smooth variety of minimal degree if the dimension of the variety of general type is even. In this article we do show that for an algebraic surface of general type, there are no triple canonical covers of rational scrolls Y or \mathbf{P}^2 or a quadric \mathbf{Q}_2 , whether they are embedded as a variety of minimal degree or not. But if the dimension of X is odd, there are infinite families of examples of triple canonical covers, and this appears in the last section.

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1. DEFORMATIONS OF A CANONICAL TRIPLE COVER OF A PROJECTIVE BUNDLE OVER \mathbf{P}^1 OR A PROJECTIVE SPACE OR A QUADRIC HYPERSURFACE EMBEDDED BY A COMPLETE LINEAR SERIES

We work over \mathbf{C} . We will use the following

Set-up and notation:

- (1) X is a smooth variety of general type of dimension $m \geq 3$ with ample and base-point-free canonical bundle and canonical map $\varphi : X \rightarrow \mathbf{P}^N$ of degree 3.
- (2) The image Y of φ is one of the following varieties:
 - (a) a smooth projective bundle $Y = \mathbf{P}(E)$ (of dimension m) over \mathbf{P}^1 (where $E = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(-e_{m-1})$ with $0 \leq e_1 \leq \cdots \leq e_{m-1}$) embedded, as a non-degenerate variety, in \mathbf{P}^N by the complete linear series of a very ample divisor $aT + bF$, for which T is the divisor on Y such that $\mathcal{O}_Y(T) = \mathcal{O}_{\mathbf{P}(E)}(1)$ and $F = \mathbf{P}^{m-1}$ is a fiber, or
 - (b) projective space $Y = \mathbf{P}^m$ embedded, as a non-degenerate variety, in \mathbf{P}^N by the complete linear series of $\mathcal{O}_{\mathbf{P}^m}(d)$, or
 - (c) a smooth quadric hypersurface $Y = \mathbf{Q}_m$ in \mathbf{P}^{m+1} embedded, as a non-degenerate variety, in \mathbf{P}^N by the complete linear series of $\mathcal{O}_{\mathbf{Q}_m}(d)$, for which $m + d \geq 6$.
- (3) Let $i : Y \hookrightarrow \mathbf{P}^N$ denote the embedding and let $\pi : X \rightarrow Y$ denote the induced triple cover such that $\varphi = i \circ \pi$.

Remark 1.1. Under the conditions of our set-up, φ factors through a triple cover $\pi : X \rightarrow Y$ whose trace-zero module is a vector bundle \mathcal{E} of rank 2. In addition, [GP03, Theorem 4.3] says that φ is the canonical map of X if, and only if,

- (1) the triple cover π is cyclic with $\mathcal{E} = \frac{1}{2}(\omega_Y(-1)) \oplus \omega_Y(-1)$, and
- (2) $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1))) = 0$.

Observe that, since $p_g(Y) = 0$, condition (2) is equivalent to the fact that the pullback of the complete linear series of $\mathcal{O}_Y(1)$ is the complete canonical series of X .

Remark 1.2. Under the conditions of our set-up, there exists a triple cover π for which $\varphi = i \circ \pi$ is the canonical map of X if, and only if,

- (1) the linear system $|\frac{-3}{2}(\omega_Y(-1))|$ contains a smooth member, and
- (2) $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1))) = 0$.

Remark 1.3. Under the conditions of our set-up, numerical conditions which are sufficient for π to exist are:

- (1) In the case $Y = \mathbf{P}(E)$
 - (a) $a \geq 1$ and $b \geq ae_{m-1} + 1$ (for $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbf{P}(E)}(aT + bF)$ very ample), and
 - (b) $m + a$ is even, and $b + e_1 + \cdots + e_{m-1}$ is even (for $\frac{1}{2}(\omega_Y(-1))$ to exist), and
 - (c) $b \geq -(e_1 + \cdots + e_{m-2} + (1 - m - a)e_{m-1} + 2)$ (for $|\frac{-3}{2}(\omega_Y(-1))|$ base-point-free), and
 - (d) $a \leq m - 1$ (for $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1))) = 0$).
- (2) In the case $Y = \mathbf{P}^m$
 - (a) $d \geq 1$ (for $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbf{P}^m}(d)$ very ample), and
 - (b) $m + d$ is odd (for $\frac{1}{2}(\omega_Y(-1))$ to exist), and
 - (c) $d \leq m$ (for $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1))) = 0$).
- (3) In the case $Y = \mathbf{Q}_m$
 - (a) $d \geq 1$ (for $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbf{Q}_m}(d)$ very ample), and
 - (b) $m + d$ is odd (for $\frac{1}{2}(\omega_Y(-1))$ to exist), and
 - (c) $d \leq m - 2$ (for $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1))) = 0$).

In [GP03], a result there shows that there are no triple canonical covers of a smooth variety of minimal degree if the dimension of the variety of general type is even. A natural question would be to know if there are any canonical triple covers at all of say projective bundle Y over a line, such that it is embedded by the complete canonical linear series. The next result shows that for an algebraic surface of general type, there are no triple canonical covers of rational scrolls Y or \mathbf{P}^2 or a quadric \mathbf{Q}_2 per se, whether they are embedded as a variety of minimal degree or not. If we relax the condition that it is only the sublinear series of the canonical linear series, but not the whole linear series, then the following result does tell us that such exist for the class of algebraic surfaces.

Proposition 1.4. *In the case $m = 2$, of a Hirzebruch surface $Y = \mathbf{F}_e$ or $Y = \mathbf{P}^2$ or $Y = \mathbf{Q}_2$, there exist cyclic triple covers $\pi : X \rightarrow Y$ for which $\omega_X = \pi^*(\mathcal{O}_Y(1))$, but the pullback $\pi^*H^0(\mathcal{O}_Y(1))$ is a non-complete subseries of $H^0(\omega_X)$, except for the case $Y = \mathbf{P}^2$ and $d = 1$.*

Proof. We will use the conditions of Remark (1.2).

First we deal with the case $Y = \mathbf{F}_e$. We claim that $|\frac{-3}{2}(\omega_Y(-1))|$ contains a smooth member if, and only if, (1) $e \leq 3$, or (2) $(b - ae \geq e - 2$ or $b - ae = \frac{1}{3}e - 2$), if $e \geq 4$. Indeed, if $|\frac{-3}{2}(\omega_Y(-1))|$ contains a smooth member but $\frac{-3}{2}(\omega_Y(-1))$ is not base-point-free, then T is the fixed part of $|\frac{-3}{2}(\omega_Y(-1))|$ and does not intersect the mobile (base-point-free) part of $|\frac{-3}{2}(\omega_Y(-1))|$. This happens if and only if $b - ae = \frac{1}{3}e - 2$. On the other hand $\frac{-3}{2}(\omega_Y(-1))$ is base-point-free if and only if $b - ae \geq e - 2$, which always holds if $0 \leq e \leq 3$, since $b - ae \geq 1$.

Now we see that we can construct a triple cover of Y . If $b - ae \geq e - 2$, then $\frac{-3}{2}(\omega_Y(-1))$ is base-point-free and obviously non trivial, so $\frac{-3}{2}(\omega_Y(-1))$ contains smooth members. If $b - ae = \frac{1}{3}e - 2$, then T is the fixed part of $|\frac{-3}{2}(\omega_Y(-1))|$ and does not intersect the mobile (base-point-free) part of $|\frac{-3}{2}(\omega_Y(-1))|$, so $|\frac{-3}{2}(\omega_Y(-1))|$ contains smooth (non-connected) members. Thus in all cases we can choose a smooth divisor $B \in |\frac{-3}{2}(\omega_Y(-1))|$. Let $\pi : X \rightarrow Y$ be the triple cyclic cover of Y branched along B . Since B is smooth, so is X . Since $B \in |\frac{-3}{2}(\omega_Y(-1))|$, ramification formula implies that $\omega_X = \pi^*(\mathcal{O}_Y(1))$.

Since $p_g(Y) = 0$, then $\pi^*H^0(\mathcal{O}_Y(1))$ equals the complete series $H^0(\omega_X)$ (so $\varphi = i \circ \pi$) if, and only if, $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1))) = 0$.

Since $\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1)) = \frac{a-2}{2}T + \frac{b-e-2}{2}F$ and $a \geq 2$ (for a is even), then we see that $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1)))$ does not vanish, since $b - e - 2 \geq 0$ (for $b \geq ae + 1$ and $b - e$ is even).

In the case $Y = \mathbf{P}^2$ it is straightforward to see that triple covers exist, since for $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbf{P}^2}(d)$ with $d \geq 1$ odd, the linear series of $H^0(\frac{-3}{2}(\omega_Y(-1)))$ is very ample (so it contains smooth members), but $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1))) = H^0(\mathcal{O}_{\mathbf{P}^2}(\frac{d-3}{2}))$ does not vanish, unless $d = 1$.

In the case $Y = \mathbf{Q}_2$ it is also straightforward to see that triple covers exist, since for $\mathcal{O}_Y(1) = \mathcal{O}_{\mathbf{Q}_2}(a, b)$ with $a, b \geq 2$ even, the linear series of $H^0(\frac{-3}{2}(\omega_Y(-1)))$ is very ample (so it contains smooth members), but $H^0(\mathcal{O}_Y(1) \otimes \frac{1}{2}(\omega_Y(-1))) = H^0(\mathcal{O}_{\mathbf{Q}_2}(\frac{a-2}{2}, \frac{b-2}{2}))$ does not vanish. \square

The next Propositions 1.5, 1.6, 1.7 are crucial technical results that lead to the main Theorem 1.8 of the article. They deal with the vanishing of a certain Ext groups. Together with duality theorems this in turn proves first cohomology vanishing of the tangent sheaf twisted with relevant line bundles, which in turn leads to the main result. One has to be careful about the interpretations of the results, which is also a crucial part of the process.

Proposition 1.5. *Let Y be a smooth projective bundle on \mathbf{P}^1 of dimension $m \geq 3$ embedded, as a non-degenerate variety, in \mathbf{P}^N by a complete linear series. Then $\text{Ext}^1(\Omega_Y, \frac{1}{2}(\omega_Y(-1))) = 0$ and $\text{Ext}^1(\Omega_Y, \omega_Y(-1)) = 0$.*

Proof. We already now by [GGP15, Proposition 1.2] that $\text{Ext}^1(\Omega_Y, \omega_Y(-1)) = 0$. What we have to prove now is $\text{Ext}^1(\Omega_Y, \frac{1}{2}(\omega_Y(-1))) = 0$.

Since $\text{Ext}^1(\Omega_Y, \frac{1}{2}(\omega_Y(-1))) = H^1(\mathcal{T}_Y \otimes \frac{1}{2}(\omega_Y(-1)))$. We will prove $H^1(\mathcal{T}_Y \otimes \frac{1}{2}(\omega_Y(-1))) = 0$.

Let H denote the hyperplane section. Having in account the sequence

$$0 \longrightarrow \mathcal{T}_{Y/\mathbf{P}^1} \longrightarrow \mathcal{T}_Y \longrightarrow p^* \mathcal{T}_{\mathbf{P}^1} \longrightarrow 0$$

and the dual of the relative Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}(E)} \longrightarrow p^* E^* \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \longrightarrow \mathcal{T}_{Y/\mathbf{P}^1} \longrightarrow 0,$$

we only need to check the vanishings of $H^1(p^* \mathcal{T}_{\mathbf{P}^1} \otimes \frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(-\frac{1}{2}H))$, $H^1(p^* E^* \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \otimes \frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(-\frac{1}{2}H))$ and $H^2(\frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(-\frac{1}{2}H))$. By Serre duality this is equivalent to proving the vanishings of $H^{m-1}(p^* \omega_{\mathbf{P}^1} \otimes \frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(\frac{1}{2}H))$, $H^{m-1}(p^* E \otimes \mathcal{O}_{\mathbf{P}(E)}(-1) \otimes \frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(\frac{1}{2}H))$ and $H^{m-2}(\frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(\frac{1}{2}H))$. These cohomology groups are isomorphic to the cohomology groups of the push-down to \mathbf{P}^1 of the three bundles. Indeed, in the three cases, when we restrict the bundles to the fiber F ($F = \mathbf{P}^{m-1}$) we obtain line bundles or direct sum of line bundles, so the intermediate cohomology of the restriction of the bundles to F vanish. The topmost (i.e., H^{m-1}) cohomology also vanishes since the line bundles just mention have degree greater than or equal to $-\frac{m}{2}$. Now the three cohomology groups of the push-downs on \mathbf{P}^1 are 0 by dimension reasons except the group $H^{m-2}(p_*(\frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(\frac{1}{2}H)))$ when $m = 3$. In this case we do also show that $H^1(p_*(\frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(\frac{1}{2}H))) = 0$. Let $E = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-e_2)$, with $0 \leq e_1 \leq e_2$, and let T be the divisor on Y such that $\mathcal{O}_Y(T) = \mathcal{O}_{\mathbf{P}(E)}(1)$. Recall that the canonical divisor of Y is $-mT - (e_1 + e_2 + 2)F$ and H is linearly equivalent to $aT + bF$. Since H is very ample, $a > 0$ and $b > ae_2$. Then $\frac{1}{2} \omega_Y \otimes \mathcal{O}_Y(\frac{1}{2}H)$ is linearly equivalent to $\frac{a-3}{2}T + \frac{b-e_1-e_2-2}{2}F$. If $a = 1$, the vanishing is clear because the push-down bundle is 0. If $a \geq 3$ (and odd), the push-down bundle is a direct sum of line bundles and the smallest of the degrees of these line bundles is $d = \frac{-(a-3)e_2 + b - e_1 - e_2 - 2}{2}$. Because of $b > ae_2$, $d > \frac{2e_2 - e_1 - 2}{2}$. Since $e_1 \leq e_2$, $d > \frac{e_2 - 2}{2}$. Finally, since $e_2 \geq 0$, $d > -1$, so H^1 of all the line bundles vanishes. \square

Proposition 1.6. *Let $Y = \mathbf{P}^m$, with $m \geq 3$, embedded, as a non-degenerate variety, in \mathbf{P}^N by the complete linear series of $\mathcal{O}_{\mathbf{P}^m}(d)$, for which $d \geq 1$ and $m + d$ is odd. Then $\text{Ext}^1(\Omega_Y, \frac{1}{2}(\omega_Y(-1))) = 0$ and $\text{Ext}^1(\Omega_Y, \omega_Y(-1)) = 0$.*

Proof. The proof is straightforward using the Euler sequence in \mathbf{P}^m . \square

Proposition 1.7. *Let $Y = \mathbf{Q}_m$, with $m \geq 3$, embedded, as a non-degenerate variety, in \mathbf{P}^N by the complete linear series of $\mathcal{O}_{\mathbf{Q}_m}(d)$, for which $d \geq 1$ and $m + d$ is odd. Then $\text{Ext}^1(\Omega_Y, \omega_Y(-1)) = 0$ and, if $m + d \geq 6$, then also $\text{Ext}^1(\Omega_Y, \frac{1}{2}(\omega_Y(-1))) = 0$.*

Proof. The proof is also straightforward using the Euler sequence in \mathbf{P}^{m+1} restricted to \mathbf{Q}_m and the normal sequence of \mathbf{Q}_m in \mathbf{P}^{m+1} . \square

The previous steps yield the main Theorem.

Theorem 1.8. *Let X, φ and Y be as in the set-up. Then any general deformation of φ is a canonical map and a finite morphism of degree 3 onto its image, which is a deformation of Y .*

Proof. We are assuming $\varphi : X \rightarrow \mathbf{P}^N$ is the canonical map of X which factors $\varphi = i \circ \pi$, for which $\pi : X \rightarrow Y$ is a triple cover onto Y , which fits in one of the three cases in our set-up and the embedding $i : Y \hookrightarrow \mathbf{P}^N$ is non-degenerate and given by a complete linear series as in the set-up. By Propositions 1.5, 1.6, 1.7, we know that $H^1(\mathcal{T}_Y \otimes \mathcal{E}) = 0$.

Let (\mathfrak{X}, Φ) be a deformation of (X, φ) over a smooth curve T and let (X', φ') be a general member of (\mathfrak{X}, Φ) .

Since $H^1(\mathcal{T}_Y \otimes \mathcal{E}) = 0$, then [Weh86, Corollary 1.11] (or [Hor76, Theorem 8.1]) tells us that \mathfrak{X} can be realized as a covering $\Pi : \mathfrak{X} \rightarrow \mathfrak{Y}$ (which, after shrinking T , is finite, surjective and of degree 3) of a deformation \mathfrak{Y} of Y . Now, since $H^1(\mathcal{T}_{\mathbf{P}^N}|_Y) = 0$ (as can be easily checked in all cases of our set-up), any deformation \mathfrak{Y} of Y can be realized as a deformation (\mathfrak{Y}, i) of (Y, i) (see [Hor76, Theorem 8.1]). Since i is an embedding, we can assume, after shrinking T , that i is a relative embedding of \mathfrak{Y} in \mathbf{P}_T^N . Then $i \circ \Pi = \Phi'$ is a deformation of φ . Now we saw in [GGP10, Lemma 2.4] that the only deformation of φ is the (relative) canonical morphism, so φ' is the canonical morphism of X' and, after shrinking T , we have $\Phi = i \circ \Pi$. \square

2. EXAMPLES AND FURTHER REMARKS

Let $\pi : X \rightarrow Y$ be a cyclic triple cover of a variety of minimal degree Y . In [GP03] it is shown that if the dimension of X is odd, then the degree of Y cannot be an even integer. Now assume that the dimension of X is even, it is shown in [GP03] that there are no canonical, generically finite morphisms of degree 3 whose image is a smooth rational scroll.

It is good to know what kind of triple covers does occur in mathematical nature. In [GP03] it is shown that triple canonical covers of \mathbf{P}^m , a hyperquadric and a smooth rational normal scroll do actually occur in many cases. We construct the covers below (see Remark 1.3).

To construct examples of canonical morphisms $\varphi : X \rightarrow \mathbf{P}^N$ of degree 3, such that φ induces a triple cover onto a smooth rational normal scroll Y of odd dimension $m \geq 3$ and even degree r , we carry out the following construction: let $Y = \mathbf{P}(E)$ over \mathbf{P}^1 , where $E = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(-e_{m-1})$, with $0 \leq e_1 \leq \cdots \leq e_{m-1}$, let T be the divisor on Y such that $\mathcal{O}_Y(T) = \mathcal{O}_{\mathbf{P}(E)}(1)$ and let $H = T + bF$ be the hyperplane divisor on Y , where F is the class of a fiber and $b - e_{m-1} \geq 1$. Then take the triple cyclic cover $\pi : X \rightarrow Y$ branched along a smooth divisor linearly equivalent to $3(\frac{m+1}{2}H - \frac{r-2}{2}F)$ (a sufficient condition for such a smooth divisor to exist is $(m+1)(b - e_{m-1}) \geq r - 2$). In this case $\omega_X = \pi^* \mathcal{O}_Y(H)$ and $H^0(\omega_X) = \pi^* H^0(\mathcal{O}_Y(H))$, which shows that $\varphi = i \circ \pi$, where $i : Y \hookrightarrow \mathbf{P}^N$ is the embedding, as wanted.

To construct examples of canonical morphisms $\varphi : X \rightarrow \mathbf{P}^N$ of degree 3, such that φ induces a triple cover onto $Y = \mathbf{P}^m$, with $m \geq 3$, take the triple cyclic cover $\pi : X \rightarrow \mathbf{P}^m$ branched along a smooth divisor of degree $\frac{3(m+d+1)}{2}$, for which $m+d$ is odd and $1 \leq d \leq m$. In this case $\omega_X = \pi^* \mathcal{O}_{\mathbf{P}^m}(d)$ and $H^0(\omega_X) = \pi^* H^0(\mathcal{O}_{\mathbf{P}^m}(d))$, which shows that $\varphi = i \circ \pi$, where $i : \mathbf{P}^m \hookrightarrow \mathbf{P}^N$ is the embedding, as wanted.

To construct examples of canonical morphisms $\varphi : X \rightarrow \mathbf{P}^N$ of degree 3, such that φ induces a triple cover onto a hyperquadric $Y = \mathbf{Q}_m$, with $m \geq 3$, take the triple cyclic cover $\pi : X \rightarrow \mathbf{Q}_m$ branched along a smooth divisor in \mathbf{Q}_m which is the complete intersection of \mathbf{Q}_m and a hypersurface in \mathbf{P}^{m+1} of degree $\frac{3(m+d-1)}{2}$, for which $m+d$ is odd and $1 \leq d \leq m-2$. In this case $\omega_X = \pi^* \mathcal{O}_{\mathbf{Q}_m}(d)$

and $H^0(\omega_X) = \pi^* H^0(\mathcal{O}_{\mathbf{Q}_m}(d))$, which shows that $\varphi = i \circ \pi$, where $i : \mathbf{Q}_m \hookrightarrow \mathbf{P}^N$ is the embedding, as wanted.

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